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Binomial Functions and Combinatorial Mathematics

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1. INTRODUCTION

Binomial functions appear in various walks of mathematics and are closely related to some combinatorial entities. In this paper we will define binomial functions, consider examples, develop some properties, and then use this development to establish some combinatorial identities and a conjecture. We will conclude with an epilogue that establishes a relationship between this paper and the papers of Rota–Mullin [14], Garsia [6], and Fillmore–Williamson [5].

The author wants to thank Professor Leonard Carlitz for communicating that he had found a generalization of one of her identities in [12] (which appears as Formula (4.20) in this paper), for it was that communication and the subsequent awareness of its relation to a combinatorial identity that motivated this paper.

2. PRELIMINARIES

We will let

$$n \text{ represent a positive integer} \tag{2.1}$$

$$c, d, h, j, k, m, r \text{ represents non-negative integers} \tag{2.2}$$

$$b, \alpha \text{ represent complex numbers} \tag{2.3}$$

$$0^0 = 1 \tag{2.4}$$

$$\binom{c}{k} \equiv \frac{c!}{k!(c-k)!} \tag{2.5}$$

when $0 \leqslant k \leqslant c$, and zero otherwise [and hence is the binomial coefficient that represents the number of ways of selecting k objects from c objects without regard to order].

$$k! \binom{m}{k} = \sum_{c=0}^{k} S_c{}^k m^c \tag{2.6}$$

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where S_c^k is a *Stirling number of the first kind* and $(-1)^{k-c} S_c^k$ is the number of permutations of k symbols that have exactly c cycles.

$$m^k = \sum_{c=0}^{k} s_c^k c! \binom{m}{c} \tag{2.7}$$

where s_c^k is a *Stirling number of the second kind* and is the number of ways of partitioning a set of k elements into c nonempty subsets. Also, $c! s_c^k$ is the number of surjective mappings of a set with k elements onto a set with c elements.

We note that symbols for the Stirling numbers have never been standardized. In fact, Abramowitz and Stegun [1, p. 822] lists nine different sets and the ones we are using form a tenth set. Our justification for these symbols is their convenient generalization in Section 4 as well as their consistency with other symbols in this paper.

The following well-known results involving the above symbols will now be listed in forms that will be useful for our development in Section 3.

$$\frac{(c+1)^k}{k!} = \sum_{j=0}^{k} \frac{c^j}{j!} \frac{1^{k-j}}{(k-j)!} \tag{2.8}$$

$$\binom{c+1}{k} = \sum_{j=0}^{k} \binom{c}{j} \binom{1}{k-j} \tag{2.9}$$

$$(c+1)! \frac{S_{c+1}^k}{k!} = \sum_{j=0}^{k} c! \frac{S_c^j}{j!} 1! \frac{S_1^{k-j}}{(k-j)!} \tag{2.10}$$

$$(c+1)! \frac{s_{c+1}^k}{k!} = \sum_{j=0}^{k} c! \frac{s_c^j}{j!} 1! \frac{s_1^{k-j}}{(k-j)!}\,. \tag{2.11}$$

Finally, we will sometimes want to count the number of ways in which k can be expressed as the sum of c non-negative (or positive) integers, where expressions with different orderings are counted as "different" representations. Thus, the number of *representations* of k as the sum of c non-negative (or positive) integers is the number of ways in which k can be expressed as the sum of c non-negative (or positive) integers, where two representations are the same iff all of the corresponding terms are the same. Hence,

$$\binom{k+c-1}{c-1} \tag{2.12}$$

is the number of representations of k as the sum of c non-negative integers, and

$$\binom{k-1}{c-1} \tag{2.13}$$

is the number of representations of k as the sum of c positive integers.

We note that (2.12) represents the number of ways of placing k balls in c boxes, when empty boxes are permitted; and (2.13) represents the number of ways of placing k balls in c boxes, when no box is permitted to be empty.

3. Binomial Functions

We shall call $B_k(c)$ a *binomial function* if

$$B_k(c+1) = \sum_{j=0}^{k} B_j(c)\, B_{k-j}(1). \tag{3.1}$$

It is easy to see that the zero function, $B_k(c) \equiv 0$, is a trivial binomial function, and henceforth we shall exclude it from our discussion. With this agreement it follows that

$$B_k(0) = 0^k \tag{3.2}$$

whenever $B_k(c)$ is a binomial function.

By virtue of Formulas (2.8) through (2.11) we can observe that

$$\frac{c^k}{k!} \tag{3.3}$$

$$\binom{c}{k} \tag{3.4}$$

$$c!\,\frac{S_c{}^k}{k!} \tag{3.5}$$

$$c!\,\frac{s_c{}^k}{k!} \tag{3.6}$$

are all binomial functions.

We will now use formal power series to establish the following basic result for binomial functions. (The proof is essentially that given by Chernoff and Waterhouse in [4].)

THEOREM 3.1. *If $B_k(c)$ is a binomial function, then*

$$B_k(c+d) = \sum_{j=0}^{k} B_j(c)\, B_{k-j}(d). \tag{3.7}$$

Proof. Let $g(c, t) = \sum_{k=0}^{\infty} B_k(c)\, t^k$ be a formal power series (so that it can be manipulated as if it represented an entire function of t). Then,

$$g(c+1, t) = \sum_{k=0}^{\infty} B_k(c+1)\, t^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} B_j(c)\, B_{k-j}(1)\, t^k$$

$$= \sum_{j=0}^{\infty} B_j(c)\, t^j \sum_{r=0}^{\infty} B_r(1)\, t^r$$

where $k - j = r$. Hence, $g(c+1, t) = g(c, t)\, g(1, t)$, and by induction we have $g(k, t) = [g(1, t)]^k$ and therefore $g(c+d, t) = g(c, t)\, g(d, t)$. Thus,

$$\sum_{k=0}^{\infty} B_k(c+d)\, t^k = \sum_{j=0}^{\infty} B_j(c)\, t^j \sum_{r=0}^{\infty} B_r(d)\, t^r$$

$$= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} B_j(c)\, B_{k-j}(d)\right] t^k,$$

where $j + r = k$; and comparing the coefficients of t^k yields the desired result.

The above theorem together with Formulas (2.9), (2.10), and (2.11) respectively yield the Vandermonde convolution,

$$\binom{c+d}{k} = \sum_{j=0}^{k} \binom{c}{j}\binom{d}{k-j}, \tag{3.8}$$

the identity

$$\binom{c+d}{c} S_{c+d}^{k} = \sum_{j=0}^{k} \binom{k}{j} S_c^{\,j} S_d^{k-j}, \tag{3.9}$$

and the modification of (3.9) which is obtained when each S is replaced by s.

In order to learn more about binomial functions, it will be useful to have the following algorithm.

THEOREM 3.2. *If $B_k(c)$ is a binomial function and $B_j(1) = \beta_j$, then*

$$B_k(c) = \sum_{*} \binom{c}{j_0\, j_1 \cdots j_k} \beta_0^{j_0} \beta_1^{j_1} \cdots \beta_k^{j_k} \tag{3.10}$$

*where * means that the sum is taken over all non-negative integers j_m such that*

$$\sum_{m=0}^{k} j_m = c \qquad \text{and} \qquad \sum_{m=0}^{k} m j_m = k;$$

and hence the multinomial coefficient

$$\binom{c}{j_0 j_1 \cdots j_k} = \frac{c!}{j_0! j_1! \cdots j_k!}$$

is the number of representations of k as the sum of c non-negative integers consisting of j_0 zeros, j_1 ones,..., and j_k k's.

Proof. In order to establish (3.10) by finite induction we note that it holds when $c = 0$ [by (3.2)] and proceed to show that it holds for $B_k(c + 1)$ whenever it holds for $B_0(c)$, $B_1(c)$,..., $B_k(c)$.

By the inductive assumption we have

$$B_k(c + 1) = \sum_{h=0}^{k} \sum_{*} \binom{c}{j_0 j_1 \cdots j_h} \beta_0^{j_0} \beta_1^{j_1} \cdots \beta_h^{j_h} \beta_{k-h}$$

where * means that the sum is taken over all non-negative integers j_m such that

$$\sum_{m=0}^{h} j_m = c \qquad \text{and} \qquad \sum_{m=0}^{h} m j_m = h;$$

and

$$\binom{c}{j_0 j_1 \cdots j_h}$$

represents the number of times that the term $\beta_0^{j_0} \beta_1^{j_1} \cdots \beta_h^{j_h}$ appears in the expansion of $B_h(c)$, as well as the number of representations of h as the sum of c non-negative integers consisting of j_0 zeros, j_1 ones,..., and j_h h's. Since each representation of k is obtained from a representation of h by adding $k - h$, and all of these representations are different, it now follows that

$$B_k(c + 1) = \sum_{**} \binom{c + 1}{J_0 J_1 \cdots J_k} \beta_0^{J_0} \beta_1^{J_1} \cdots \beta_k^{J_k}$$

when $J_{k-h} = 1 + j_{k-h}$, $J_m = j_m$ if $m \neq k - h$; and ** means that the sum is taken over all non-negative integers J_m such that

$$\sum_{m=0}^{k} J_m = c + 1 \qquad \text{and} \qquad \sum_{m=0}^{k} m J_m = k$$

because the term $\beta_0^{J_0} \beta_1^{J_1} \cdots \beta_k^{J_k}$ appears once for each representation of k as the sum of $c + 1$ non-negative integers consisting of J_0 zeros, J_1 ones,..., and J_k k's and each of these representations occurs exactly once. The proof by induction is now complete.

From the proof of the above theorem, we get the following.

THEOREM 3.3. *Let $\beta_0, \beta_1, \ldots, \beta_k$ be any elements in a ring and let $B_j(1) = \beta_j$ for $j = 0, 1, \ldots, k$. Then the $B_k(c)$ given by Formula* (3.10) *is a binomial function.*

The above theorems can be used in various ways. For example, Theorem 3.2 indicates that the binomial function $B_k(c)$ is completely determined by its values at $c = 1$, and hence the binomial functions (3.3) through (3.6) are characterized (respectively) by

$$\beta_k = \frac{1}{k!} \tag{3.3'}$$

$$\beta_k = \binom{1}{k} \tag{3.4'}$$

$$\beta_0 = 0, \qquad \beta_n = \frac{(-1)^{n-1}}{n} \tag{3.5'}$$

$$\beta_0 = 0, \qquad \beta_n = \frac{1}{n!}. \tag{3.6'}$$

Also, Theorem 3.3 can be used to establish an unlimited number of binomial functions, since the β_j can be chosen arbitrarily. In particular, if $\beta_j \equiv 1$, Formula (2.12) provides that

$$B_k(c) = \binom{k+c-1}{c-1}; \tag{3.11}$$

and if $\beta_0 = 0$ and $\beta_n \equiv 1$, then Formula (2.13) provides that

$$B_k(c) = \binom{k-1}{c-1}. \tag{3.12}$$

We note that the binomial function in (3.11) yields the well-known combinatorial identity

$$\binom{k+c}{c} = \sum_{j=0}^{k} \binom{j+c-1}{c-1}. \tag{3.13}$$

A similar result can be obtained from (3.12).

In order to obtain more applications of the above theorems, we will need to manufacture more examples.

4. MORE EXAMPLES

All of our examples of binomial functions have been directly or indirectly related to powers. It turns out that some interesting binomial functions can be found among the various functions related to generalized powers [11]. Although generalized powers and their properties are developed in [11] and [12], we will now included some basic definitions and results for the sake of completeness.

We first let

$$\{0\}_b \equiv 1 \tag{4.1}$$

and

$$\{n\}_b \equiv 1(1+b)(1+b+b^2)\cdots(1+b+\cdots+b^{n-1}) \tag{4.2}$$

Since $\{n\}_1 = n!$ and $\{0\}_1 = 0!$, we have

$$\{k\}_1 = k! \tag{4.3}$$

and therefore $\{k\}_b$ is a *generalized factorial.*

Note. Henceforth, we will let $\{k\}_b = \{k\}$ unless the b needs to be emphasized.

We will now use this generalized factorial to obtain the *generalized binomial coefficient*

$$N_h{}^k(b) \equiv \frac{\{k\}}{\{h\}\{k-h\}}$$

when $0 \leqslant h \leqslant k$, and zero otherwise; and observe that $N_h{}^k(1) = \binom{k}{h}$. It turns out that $N_h{}^k(b)$ is a polynomial in b (if we assume continuity when b is a proper root of unity) and hence we can call it an *N-polynomial.* (It has also been called a "Gaussian polynomial" and "q-binomial coefficient" (when q replaces b) and its properties have played important roles in several papers including [2, 3, 7, 11].) This N-polynomial has some well-established combinatorial interpretations. For example, if b is the power of a prime, then $N_h{}^k(b)$ is the number of h-dimensional subspaces in a k-dimensional linear space over $GF(b)$, the field with b elements.

We can now give a recursive definition for generalized powers by letting

$$0^k_{(b)} \equiv 0^k \tag{4.4}$$

and

$$(c+1)^k_{(b)} \equiv \sum_{h=0}^{k} N_h{}^k(b)\, c^h_{(b)}\,. \tag{4.5}$$

We note that $0^k_{(1)} = 0^k$, $1^k_{(1)} = 1_{(1)}$ $2^k_{(1)} = 2^k, \ldots$, and, by induction,

$$c^k_{(1)} = c^k. \tag{4.6}$$

Hence, it is legitimate to call $c^k_{(b)}$ a *generalized power*.

It is easy to show that Definitions (4.4) and (4.5) yield

$$1^k_{(b)} = 1 \tag{4.7}$$

and hence from our earlier observation on a combinatorial interpretation for $N_h^k(b)$, $2^k_{(b)}$ represents the number of subspaces in a k-dimensional linear space over $GF(b)$.

We can now write (4.5) as

$$\frac{(c+1)^k_{(b)}}{\{k\}} = \sum_{h=0}^{k} \frac{c^h_{(b)}}{\{h\}} \frac{1^{k-h}_{(b)}}{\{k-h\}} \tag{4.8}$$

and therefore

$$\frac{c^k_{(b)}}{\{k\}_b} \quad \text{is a binomial function with} \quad \beta_j = \frac{1}{\{j\}_b}. \tag{4.9}$$

Theorem 3.1 can now be used to obtain

$$(c+d)^k_{(b)} = \sum_{h=0}^{k} N_h{}^k(b)\, c^h_{(b)}\, d^{k-h}_{(b)}, \tag{4.10}$$

a generalized binomial formula which reduces to the ordinary one when $b = 1$.

Volk's generalization of (4.8) in [15] provides a binomial function that yields a more general generalized binomial formula.

Since $\{k\}_0 = 1$, it follows from (4.9) that $c^k_{(0)}$ is a binomial function with $\beta_j = 1$. Hence, we can use (3.11) to show that

$$c^k_{(0)} = \binom{c+k-1}{k}. \tag{4.11}$$

In order to obtain more binomial functions, we observe that

$$c^k_{(b)} = \sum_{j=0}^{c-1} [(j+1)^k_{(b)} - j^k_{(b)}] = \sum_{h=0}^{k-1} N_h{}^k(b) \sum_{j=0}^{c-1} j^h_{(b)}$$

and hence, by induction, we get

$$c^k_{(b)} = \sum_{j=0}^{k} R_j{}^k(b) \binom{c}{j}, \tag{4.12}$$

where

$$R_0{}^k(b) = 0^k \tag{4.13}$$

and

$$R_j{}^k(b) = \sum_{h=0}^{k-1} R^h_{j-1}(b)\, N_h{}^k(b) \tag{4.14}$$

when $0 < j \leqslant k$ and zero otherwise. Hence, $R_j{}^k(b)$ is a polynomial in b which is the sum of products of N-polynomials.

If we let $j = 1$ in (4.14) we can use (4.13) to obtain $R_1^k(b) = 1 - 0^k$. Hence, the recurrence (4.14) yields

$$\frac{R_{c+1}^k(b)}{\{k\}} = \sum_{j=0}^{k} \frac{R_c^j(b)}{\{j\}} \frac{R_1^{k-j}(b)}{\{k-j\}} \tag{4.15}$$

and therefore

$$\frac{R_c^k(b)}{\{k\}_b} \quad \text{is a binomial function with} \quad \beta_0 = 0, \quad \beta_n = \frac{1}{\{n\}_b}. \tag{4.16}$$

Theorem 3.1 can now be used to obtain the expected generalization of (4.15).

By joint use of (4.12), (4.6), and (2.7), we observe that $(j!)^{-1} R_j^k(1) = s_j^k$, and hence

$$s_j^k(b) \equiv (j!)^{-1} R_j^k(b) \tag{4.17}$$

is a generalized *Stirling number of the second kind* with

$$s_j^k(1) = s_j^k.$$

After expanding the right-hand member of (4.12), we obtain

$$c_{(b)}^k = \sum_{j=0}^{k} P_j^k(b)\, c^j, \tag{4.18}$$

where the *P-polynomial*

$$P_j^k(b) = \sum_{m=j}^{k} S_j^m s_m^k(b) \tag{4.19}$$

when $0 \leqslant j \leqslant k$ and zero otherwise.

Hence, a P-polynomial is a linear combination of generalized Stirling numbers of the second kind with ordinary Stirling numbers of the first kind as coefficients.

In [12] it is shown that

$$(c+1)!\, \frac{P_{c+1}^k(b)}{\{k\}} = \sum_{j=0}^{k} c!\, \frac{P_c^j(b)}{\{j\}}\, 1!\, \frac{P_1^{k-j}(b)}{\{k-j\}}, \tag{4.20}$$

and that $nP_1^n(b) = (1-b)^{n-1}\{n-1\}$; and hence

$$c!\, \frac{P_c^k(b)}{\{k\}_b} \quad \text{is a binomial function with} \quad \beta_0 = 0, \quad \beta_n = \frac{(1-b)^n}{n(1-b^n)}. \tag{4.21}$$

Theorem 3.1 can now be used to obtain the expected generalization of (4.20). We note that this generalization was not "expected" by the author until it was communicated by Professor Carlitz, as stated in the introduction.

It turns out that either (4.12) or (4.18) can be used to define $\alpha^k_{(b)}$, since in each case the right-hand member is defined when c is a complex number. We will be able to conduct a more complete investigation of $\alpha^k_{(b)}$ if we know more about $R_j{}^k(b)$ and $P_j{}^k(b)$; and the binomial function algorithm, Theorem 3.2, provides a useful method for finding explicit expressions for $R^k_{k-j}(b)$ and $P^k_{k-j}(b)$ when $j = 0, 1, 2, \ldots$.

We will now conclude this section with a list of four formulas (proved in [11] and [12]) involving the binomial functions that we have been discussing.

In the next section we will show how the first three provide identities for Stirling numbers and how the last one provides an identity for binomial coefficients.

$$\sum_{m=j}^{k} s_j{}^m P_m{}^k(b) = s_j{}^k(b), \tag{4.22}$$

$$\sum_{m=j}^{k} P_j{}^m(b)\, N_m{}^k(b) = \sum_{m=j}^{k} \binom{m}{j} P_m{}^k(b), \tag{4.23}$$

$$\sum_{k=j}^{\infty} \alpha^k \frac{P_j{}^k(b)}{\{k\}_b} = \frac{[E(b, \alpha)]^j}{j!} \tag{4.24}$$

where α is a complex number such that $|\alpha| < 1$ and $E(0, \alpha) = -\mathrm{Log}(1 - \alpha)$ with Log representing the principal value of the logarithm so that its imaginary part lies in the interval $(-\pi, \pi]$,

$$(-1)^j j! s_j{}^k(b) = \sum_{m=0}^{j} (-1)^m \binom{j}{m} m^k_{(b)} . \tag{4.25}$$

Since all of the above identities involve the variable b, we shall call them "b-identities."

5. From 0-Identities to Combinatorial Identities

In this section we will show how various b-identities involving binomial functions can be converted to combinatorial identities by letting $b = 0$. This feature stems from the fact that

$$c^k_{(0)} = (-1)^k \binom{-c}{k}, \tag{5.1}$$

$$s_j{}^k(0) = \frac{1}{j!} \binom{k-1}{j-1}, \tag{5.2}$$

and

$$P_j^k(0) = (-1)^{j+k} \frac{S_j^k}{k!}. \tag{5.3}$$

Formula (5.1) follows from (4.11); (5.2) follows from (4.17) and (3.12) since $R_j^k(0)$ is a binomial function with $\beta_0 = 0$, $\beta_n = 1$; and (5.3) follows from (3.5′) since

$$(-1)^{c+k}\, c! P_c^k(0)$$

is a binomial function with $\beta_0 = 0$, $\beta_n = (-1)^{n-1}/n$. [It also follows directly from (5.1) and (4.18).]

We will now proceed to use the above formulas together with

$$N_j^k(0) = \{k\}_0 = 1 \tag{5.4}$$

when $0 \leqslant j \leqslant k$ to obtain combinatorial identities from 0-identities.

First, by use of (5.3) we can obtain (2.10) from (4.20). Now, letting $b = 0$ in Formulas (4.19), (4.22), (4.23), (4.24), (4.25) respectively yield

$$(-1)^{j+k} S_j^k = \sum_{m=j}^{k} \frac{k!}{m!} \binom{k-1}{m-1} S_j^m \tag{5.5}$$

$$\sum_{m=j}^{k} (-1)^m s_j^m S_m^k = (-1)^k \frac{k!}{j!} \binom{k-1}{j-1} \tag{5.6}$$

$$\sum_{m=j}^{k} (-1)^{m+j} \frac{S_j^m}{m!} = \sum_{m=j}^{k} (-1)^{m+k} \binom{m}{j} \frac{S_m^k}{k!} \tag{5.7}$$

$$\sum_{k=j}^{\infty} \alpha^k \frac{S_j^k}{k!} = \frac{[\operatorname{Log}(1+\alpha)]^j}{j!} \quad \text{if} \quad |\alpha| < 1 \tag{5.8}$$

$$(-1)^{j+k} \binom{k-1}{j-1} = \sum_{m=0}^{j} (-1)^m \binom{j}{m} \binom{-m}{k} \tag{5.9}$$

with the aid of (5.1), (5.2), and (5.3).

We note that Identities (5.6), (5.8), and (5.9) appear in Riordan [13, pp. 44, 42], and Gould [9, p. 27 (3.47)] respectively. Also, by applying the techniques described in Andrews [2, Sect. 5], hypergeometric series can be used to show that (5.9) is equivalent to the Vandermonde convolution (3.8); and the author wants to thank the referee for calling this fact to her attention.

We also note that the right-hand member of (5.6) is a *Lah number* [10] and therefore the right-hand member of (5.5) is a linear combination of Lah numbers.

6. A Conjecture

If

$$A_c^k = c!\,\frac{S_c^k}{k!} \qquad \text{and} \qquad B_c^k = c!\,\frac{s_c^k}{k!},$$

then

$$\binom{m}{k} = \sum_{c=0}^{k} A_c^k \frac{m^c}{c!} \qquad \text{and} \qquad \frac{m^k}{k!} = \sum_{c=0}^{k} B_c^k \binom{m}{c},$$

and hence (assuming $c \leqslant k$) we have the orthogonal relation

$$\sum_{r=c}^{k} A_c^r B_r^k = \sum_{r=c}^{k} B_c^r A_r^k = 0^{k-c}. \tag{6.1}$$

Since both A_c^k and B_c^k are binomial functions, one may suspect the following.

Conjecture. If $B_c^k = B_k(c)$ is a binomial function such that $B_1^0 = 0$ and $B_1^1 \neq 0$, then there exists a unique binomial function $A_c^k = A_k(c)$ such that (6.1) holds.

It is easy to establish the uniqueness. We have used brute force to try to establish the binomiality and now tend to believe that it can be proved by means of a simple device that has escaped our notice.

7. Epilogue

Since writing the previous six sections, I have become aware of a connection between this work and "polynomials of binomial type" as presented by Rota–Mullin [14], Garsia [6], Fillmore–Williamson [5], and others—thanks to an anonymous referee as well as discussions with Professors George Andrews, Richard Askey, Ed Bender, Adriano Garsia, and Gill Williamson. We will now indicate how some of the concepts and terminology in those papers relate to the above.

We first note that the polynomial $B_k(c)$ is a binomial function iff $k!B_k(c)$ is a sequence of polynomials of binomial type [by Theorem 3.1]. Hence, all examples of binomial functions that are also polynomials can be converted to polynomials of binomial type by means of a simple "*de*-normalization." It may be noted that some of the binomial functions in this paper [i.e., those given in (3.5), (3.6), (4.16), and (4.21)] are not polynomials, for each of them has $\beta_0 = 0$ and hence is zero when $c > k$.

We will now indicate how some of our results are related to the theory of polynomials of binomial type (which will be referred to as PBT).

Since [12, p. 771]

$$\sum_{k \geqslant 0} \alpha^k \frac{c_{(b)}^k}{\{k\}} = \mathrm{Exp}_b(\alpha \cdot c) = e^{cE(b,\alpha)}, \tag{7.1}$$

it follows from (4.9), (4.18), and PBT that $E_b(\alpha) = E(b, \alpha)$ is a power series in α in which the constant term is zero and the coefficient of α is different from zero (this fact can be verified in [12, p. 768]) and that $E_b^{-1}(D)$ is the "delta operator" corresponding to the PBT

$$p_k(x) = k! \frac{x_{(b)}^k}{\{k\}} \tag{7.2}$$

when D is the differentiation operator.

Since the linear operator $D_b{}^1$ (as defined in [12, p. 777]) has the property

$$D_b{}^1 p_k(x) = k p_{k-1}(x), \tag{7.3}$$

we see that $D_b{}^1$ is the "delta operator" for p_k, and so we have

$$D_b{}^1 = E_b^{-1}(D). \tag{7.4}$$

Hence, PBT provides a method for finding $D_b{}^1$ as a power series in D.

It can be shown that

$$\sum_{k \geqslant c} \alpha^k c! \frac{P_c{}^k(b)}{\{k\}} = [E(b, \alpha)]^c \tag{7.5}$$

(as in [12, p. 772]) and

$$\sum_{k \geqslant c} \alpha^k \frac{R_c{}^k(b)}{\{k\}} = [\mathrm{Exp}_b(\alpha \cdot 1) - 1]^c \tag{7.6}$$

since

$$\mathrm{Exp}_b(\alpha \cdot x) = \sum_{c \geqslant 0} [\mathrm{Exp}_b(\alpha \cdot 1) - 1]^c \binom{x}{c}. \tag{7.7}$$

We observe that PBT can be used to show that the binomial functions

$$c! \frac{P_c{}^k(b)}{\{k\}} \quad \text{and} \quad \frac{R_c{}^k(b)}{\{k\}}$$

are not polynomials in c since the right-hand members of (7.5) and (7.6) are not expressible as $e^{cg(\alpha)}$ where $g(\alpha)$ is a power series in α (because when $\alpha = 0$, we have $E(b, \alpha) = 0$ and $\mathrm{Exp}_b(\alpha \cdot 1) = 1$).

Finally, we note that an application of the device used in Garsia [6, pp. 64, 65] can be used to establish the following result.

THEOREM 7.1. *If (using the symbol established in* [12, p. 761])

$$\mathrm{Exp}_b[g(\alpha)\cdot x] = \sum_{k\geqslant 0} \frac{p_k(x, b)}{\{k\}}\,\alpha^k$$

where $g(\alpha) = \sum_{n\geqslant 1} a_n\alpha^n$, $a_1 \neq 0$, *then* $k!(p_k(x, b)/\{k\})$ *is a polynomial of binomial type whose "delta operator" is* $g^{-1}(D_b{}^1)$.

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